Fluctuations and transients in quantum-resonant evolution

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The quantum-resonant evolution of the mean kinetic energy (MKE) of the kicked particle is studied in detail on different time scales for general periodic kicking potentials. It is shown that the asymptotic time behavior of a wave-packet MKE is typically a linear growth with bounded fluctuations having a simple number-theoretical origin. For a large class of wave packets, the MKE is shown to be exactly the superposition of its asymptotic behavior and transient logarithmic corrections. Both fluctuations and transients can be significant for not too large times but they may vanish identically under some conditions. In the case of incoherent mixtures of plane waves, it is shown that the MKE never exhibits asymptotic fluctuations but transients usually occur.

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The recent extensive studies of quantum resonance (QR) and related phenomena exhibited by the periodically kicked particle (KP), either in the presence or in the absence of gravity, have led to interesting new connections between the classical and quantum dynamics of nonintegrable systems [1–4]. These phenomena are now recognized as highly efficient methods for imparting large changes in the kinetic energy of cold atoms in atom-optics experiments [1,2]. The quantum KP in the absence of gravity is described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2} + kV(\hat{x})\sum_t \delta(t' - t\tau), \qquad (1)$$

where $(\hat{x}, \hat{p})$ are position and momentum operators, k is the nonintegrability parameter, $V(x)$ is a periodic potential, t' and t are the continuous and "integer" times, and τ is the kicking period; the units are chosen so that the particle mass is 1, $\hbar = 1$, and the period of $V(x)$ is 2π. The most investigated case of QR, which is also the easiest to observe experimentally, corresponds to integer values of $\tau/(2\pi)$. It was shown [3–5] that in this case QR manifests itself in an asymptotic linear growth, as a function of time t, of the mean kinetic energy (MKE) of either a general KP wave packet or an incoherent mixture of plane waves.

Experimental observations of the QR phenomenon are, of course, limited in time. In addition, since integer values of $\tau/(2\pi)$ cannot be precisely realized experimentally, the growth of the MKE may stop or saturate after some finite time. This was shown in Ref. [4] when studying the evolution of an incoherent mixture of plane waves for $\tau = 2\pi + \epsilon$ ($|\epsilon| \ll 1$), using the standard potential $V(x) = \cos(x)$. It was found that the MKE $\langle E_{t,\epsilon} \rangle$ of this mixture essentially coincides with that for $\epsilon = 0$ in the time interval $0 < t < |k\epsilon|^{-1/2}$, where $\langle E_{t,\epsilon} \rangle \approx \langle E_{t,0} \rangle$ is already approximately given by the asymptotic linear behavior of $\langle E_{t,0} \rangle$; however, for $t > |k\epsilon|^{-1/2}$, the growth of $\langle E_{t,\epsilon} \rangle$ is suppressed due to dynamical localization. In Ref. [5], the momentum probability distribution of a wave packet for a two-harmonic potential was shown to be also robust, on some initial time interval, under small deviations ϵ of τ around $\tau = 2\pi$. These facts motivate the investigation of quantum-resonant evolution on arbitrary time scales, not just on asymptotic ones. In particular, since

the asymptotic behavior may start in practice at times t not too long, it is important to inquire about the exact nature of this behavior, i.e., whether fluctuations occur around the (average) linear growth for general periodic potentials $V(x)$ and what is the strength of these fluctuations.

In this paper, the quantum-resonant evolution of the KP is studied in detail on different time scales for integer $\tau/(2\pi)$ and general $V(x)$. In the case of the MKE $\langle E \rangle_t$ of wave packets, we show that asymptotic fluctuations of $\langle E \rangle_t$ take place only if $V(x)$ is multiharmonic and the initial wave packet is nonuniform (in a well-defined sense). These fluctuations and the average linear growth have essentially the same origin, of a simple number-theoretical nature. The fluctuations have bounded variation in time and are therefore much smaller than $\langle E \rangle_t$ for large t. For a large class of initial wave packets, we derive exact closed expressions for $\langle E \rangle_t$: $\langle E \rangle_t$ is just the superposition of its asymptotic behavior (linear growth and fluctuations) and transient logarithmic [$\ln(t)$] corrections. The fluctuations are exactly periodic or quasiperiodic in time and both they and the transients can be significant for not too large times. The transients vanish if $V(x)$ and the initial wave packet satisfy some symmetry conditions. Then, the asymptotic behavior starts at $t = 1$. If also the fluctuations vanish, e.g., for monoharmonic $V(x)$, $\langle E \rangle_t$ is an exactly linear function of t. In the case of incoherent mixtures of plane waves, we show that the MKE never exhibits asymptotic fluctuations but transients occur for nonuniform mixtures.

We first summarize known facts [3–5] about the KP quantum dynamics. The one-period evolution operator for (1), from $t' = t + 0$ to $t' = t + \tau + 0$, is given by

$$\hat{U} = \exp[-ikV(\hat{x})]\exp(-i\tau\hat{p}^2/2) \qquad (2)$$

($\hbar = 1$). Because of the 2π-periodicity of (2) in $\hat{x}$, it is natural to study the quantum evolution in terms of Bloch functions $\varphi_\beta(x) = \exp(i\beta x)\psi_\beta(x)$, where β is a quasimomentum, $0 \leq \beta < 1$, and $\psi_\beta(x + 2\pi) = \psi_\beta(x)$. In fact, the application of $\hat{U}$ on $\varphi_\beta(x)$ results in a Bloch function associated with the same quasimomentum, $\hat{U}\varphi_\beta(x) = \exp(i\beta x)\psi'_\beta(x)$; here $\psi'_\beta(x)$ is the 2π-periodic function $\psi'_\beta(x) = \hat{U}_\beta\psi_\beta(x)$, where

$$\hat{U}_\beta = \exp[-ikV(\hat{x})]\exp[-i\tau(\hat{p}+\beta)^2/2].$$ (3)

Since the operator (3) acts only on 2π-periodic functions $\psi_\beta(x)$, one can interpret x as an angle and $\hat{p}$ as an angular-momentum operator with integer eigenvalues n. Then, (3) may be viewed as the one-period evolution operator for a "β-kicked rotor" (β-KR). Given an arbitrary KP wave packet $\Psi(x)$ and using the relation between $\Psi(x)$ and its momentum representation $\widetilde{\Psi}(p)$, it is easy to see that $\Psi(x)$ can be always expressed as a superposition of Bloch functions, $\Psi(x) = \int_0^1 d\beta \exp(i\beta x)\psi_\beta(x)$, where

$$\psi_\beta(x) = \frac{1}{\sqrt{2\pi}}\sum_n \widetilde{\Psi}(n+\beta)\exp(inx).$$ (4)

One then gets the basic relation

$$\Psi_t(x) \equiv \hat{U}^t\Psi(x) = \int_0^1 d\beta \exp(i\beta x)\hat{U}_\beta^t\psi_\beta(x)$$ (5)

for integer time t, connecting the quantum dynamics of the KP with that of β-KRs. The evolution of a β-KR wave packet $\psi_\beta(x)$ under the QR condition $\tau/(2\pi)=l$, where l is a positive integer, was studied in Ref. [5] for general 2π-periodic potential $V(x)$,

$$V(x) = \sum_m V_m \exp(-imx).$$ (6)

It was found that, up to a nonrelevant constant phase factor, one has [5]

$$\psi_{\beta,t}(x) \equiv \hat{U}_\beta^t\psi_\beta(x) = \exp[-ik\bar{V}_{\beta,t}(x)]\psi_\beta(x-t\tau_\beta),$$ (7)

where $\tau_\beta = \pi l(2\beta+1)$ and

$$\bar{V}_{\beta,t}(x) = \sum_{s=0}^{t-1} V(x-s\tau_\beta) = \sum_m V_m \frac{\sin(m\tau_\beta t/2)}{\sin(m\tau_\beta/2)} e^{-im[x-(t-1)\tau_\beta/2]}.$$ (8)

The expectation value $\langle E \rangle_t$ of the kinetic energy $\hat{p}^2/2$ in the wave packet (5) can be expressed in the p representation with p decomposed into its integer (n) and fractional (β) parts, $p=n+\beta$. The resulting expression can be compactly written in terms of the Bloch functions $\varphi_{\beta,t}(x) = \exp(i\beta x)\psi_{\beta,t}(x)$, using (4) (see note [7]):

$$\langle E \rangle_t = \frac{1}{2}\int_0^1 d\beta \sum_n (n+\beta)^2|\widetilde{\Psi}_t(n+\beta)|^2$$

$$= \frac{1}{2}\int_0^1 d\beta \int_0^{2\pi} dx \left|\frac{d\varphi_{\beta,t}(x)}{dx}\right|^2.$$ (9)

Under QR conditions, $\tau/(2\pi)=l$, we see that Eq. (7) is satisfied with φ_β replacing ψ_β on both sides, up to the factor $\exp(i\beta t\tau_\beta)$. This factor, however, will not appear in (9). We then easily find that $\langle E \rangle_t$ is the sum of three terms, $\langle E \rangle_t = \langle E \rangle_t^{(1)} + \langle E \rangle_t^{(2)} + \langle E \rangle_t^{(3)}$, where

$$\langle E \rangle_t^{(1)} = \frac{k^2}{2}\int_0^1 d\beta \int_0^{2\pi} dx \left|\varphi_\beta(x)\frac{d\bar{V}_{\beta,t}(x+t\tau_\beta)}{dx}\right|^2,$$ (10)

$$\langle E \rangle_t^{(2)} = k\int_0^1 d\beta \int_0^{2\pi} dx \, \mathrm{Im}\left(\varphi_\beta(x)\frac{d\varphi_\beta^*(x)}{dx}\right)\frac{d\bar{V}_{\beta,t}(x+t\tau_\beta)}{dx},$$ (11)

$$\langle E \rangle_t^{(3)} = \frac{1}{2}\int_0^1 d\beta \int_0^{2\pi} dx \left|\frac{d\varphi_\beta(x)}{dx}\right|^2.$$ (12)

The term (12) is just a constant, the initial value $\langle E \rangle_0$ of $\langle E \rangle_t$. We now derive more explicit expressions for (10) and (11). We then show that (10) is totally responsible for the asymptotic behavior of $\langle E \rangle_t$, including characteristic fluctuations. The term (11) is shown to contribute transient effects which can be significant in some cases.

To express (10) more explicitly, we use an expansion following from Eq. (4):

$$|\varphi_\beta(x)|^2 = |\psi_\beta(x)|^2 = \frac{1}{2\pi}\sum_m C_\beta(m)\exp(imx),$$ (13)

where $C_\beta(m)$ are correlations of the KP wave packet in angular-momentum (fixed β) space,

$$C_\beta(m) = \sum_n \widetilde{\Psi}(m+n+\beta)\widetilde{\Psi}^*(n+\beta) = \sum_j \bar{C}_j(m)\exp(2\pi ij\beta).$$ (14)

The last expression in (14) is the Fourier expansion of $C_\beta(m)$, based on the obvious periodicity $C_{\beta+1}(m)=C_\beta(m)$. We insert Eqs. (13), (14), and (8) in (10) and use the identity

$$\frac{\sin(m\tau_\beta t/2)}{\sin(m\tau_\beta/2)} = \exp[im(t-1)\tau_\beta/2]\sum_{s=0}^{t-1}\exp(-ims\tau_\beta).$$ (15)

Since $\tau_\beta = \pi l(2\beta+1)$, the integrations in (10) can be explicitly performed and we finally obtain the general exact expression

$$\langle E \rangle_t^{(1)} = \frac{k^2}{2}\sum_{m,m'} mm' V_m V_{m'}^* \sum_{s,s'=1}^{t} (-1)^{l(ms-m's')} \times \bar{C}_{l(ms-m's')}(m-m').$$ (16)

Consider now the term (11). Using $\varphi_{\beta,t}(x) = \exp(i\beta x)\psi_{\beta,t}(x)$ with (4), we find that

$$\mathrm{Im}\left(\varphi_\beta(x)\frac{d\varphi_\beta^*(x)}{dx}\right) = -\frac{1}{4\pi}\sum_{n,n'}\widetilde{\Psi}(n+\beta)\widetilde{\Psi}^*(n'+\beta)(n+n' + 2\beta)\exp[i(n-n')x].$$ (17)

Let us introduce the quantity

$$G_\beta(m) = \sum_n \widetilde{\Psi}(m+n+\beta)\widetilde{\Psi}^*(n+\beta)(n+\beta)$$

$$= \sum_j \bar{G}_j(m)\exp(2\pi ij\beta),$$ (18)

where the last expression is the Fourier expansion of $G_\beta(m)$, based on the periodicity $G_{\beta+1}(m)=G_\beta(m)$. Using Eqs. (17), (8), (15), (14), and (18) in (11), we get, after a straightforward calculation,

$$\langle E \rangle_t^{(2)} = \frac{ik}{2}\sum_m V_m \sum_{s=1}^{t}(-1)^{lms}[m^2\bar{C}_{lms}(m) + 2m\bar{G}_{lms}(m)].$$ (19)

The growth of (19) with t is always slower than linear since both $\bar{C}_j(m)$ and $\bar{G}_j(m)$ decay with $|j|$. For sufficiently smooth $\widetilde{\Psi}(p)$, this decay is fast enough to yield a finite value of (19) for all t. Also, $\langle E \rangle_t^{(2)}$ may vanish identically if some symmetry conditions are satisfied. Let $\widetilde{\Psi}(p)$ have a definite parity,

$\tilde{\Psi}(-p)=\pm\tilde{\Psi}(p)$. Then, from (14) and (18), $C_\beta(m)$ $=C_{-\beta}(-m)$, i.e., $\bar{C}_j(m)=\bar{C}_{-j}(-m)$, and $G_\beta(m)=-G_{-\beta}(-m)$, i.e., $\bar{G}_j(m)=-\bar{G}_{-j}(-m)$. If also $V(x)$ is odd, $V(-x)=-V(x)$, i.e., $V_{-m}=-V_m$, it follows immediately that $\langle E\rangle_t^{(2)}=0$ in (19).

We now show that the asymptotic behavior of (16) for large t is typically a linear growth with bounded fluctuations; then, since (19) grows slower than linearly (see above), the asymptotic average growth of $\langle E\rangle_t$ is linear. Let us first change variables in the second sum in (16) from (s,s') to integer variables (a,b) defined as follows. We denote by $g=g(m,m')$ the greatest common factor of $(|m|,|m'|)$ and write $m=gm_0$, $m'=gm_0'$, where (m_0,m_0') are coprime integers; thus, $ms-m's'=g(m_0s-m_0's')$ in (16). For any integer a, a solution (s,s') of the Diophantine equation $m_0s-m_0's'$ $=a$ is $s=as_0$ and $s'=as_0'$, where (s_0,s_0') are coprime integers satisfying $m_0s_0-m_0's_0'=1$; such integers always exist for coprime (m_0,m_0'), see, e.g., Ref. [6]. The general solution (s,s') of $m_0s-m_0's'=a$ is then

$$s=as_0+bm_0', \quad s'=as_0'+bm_0 \quad (20)$$

for all integers b. By the change of variables (20), $l(ms-m's')=gla$ in (16) and, due to the decay of $\bar{C}_j(m)$ with $|j|$, $\bar{C}_{gla}(m-m')$ is negligible if $|a|$ is too large. Now, for given a, the values of b in (20) are restricted by the condition $1\le s,s'\le t$ in (16). If (m_0,m_0') have the *same* sign, the number N_b of b values for $t\gg\max(|as_0|,|as_0'|)$ is approximately $[t/\bar{m}]$, where $\bar{m}=\max(|m_0|,|m_0'|)$ and $[x]$ denotes the integer part of x. The case of m_0 and m_0' having different signs can be ignored since N_b is much smaller than $[t/\bar{m}]$ in this case. One can therefore approximate (16) for large enough t as follows:

$$\langle E\rangle_t^{(1)}\approx\frac{k^2}{2}\sum_{mm'>0}mm'V_mV_{m'}^*\sum_a(-1)^{gla}\bar{C}_{gla}(m-m')\left[\frac{t}{\bar{m}}\right].$$
$$(21)$$

We decompose $[t/\bar{m}]$ as $[t/\bar{m}]=t/\bar{m}+F_t(\bar{m})$, where $F_t(\bar{m})$ $=[t/\bar{m}]-t/\bar{m}$ are "fluctuations" of a simple number-theoretical nature. Correspondingly, $\langle E\rangle_t^{(1)}$ in (21) will be decomposed into a linear-growth part $\langle E\rangle_t^{(1,L)}$ and a fluctuating part $\langle E\rangle_t^{(1,F)}$: $\langle E\rangle_t^{(1)}\approx\langle E\rangle_t^{(1,L)}+\langle E\rangle_t^{(1,F)}$. The expression for $\langle E\rangle_t^{(1,L)}$, given by Eq. (21) with $[t/\bar{m}]$ replaced by $t/\bar{m}$, can be easily shown to be precisely the known one for the asymptotic linear growth derived in Ref. [5] [see Eqs. (17) and (19) there]. In fact, from Eq. (14) we find by simple algebra that

$$\sum_a(-1)^{gla}\bar{C}_{gla}(m-m')=\frac{1}{gl}\sum_{r=0}^{gl-1}C_{\beta_{r,g}}(m-m'), \quad (22)$$

where $\beta_{r,g}=r/(gl)-1/2\bmod(1)$, $r=0,\dots,gl-1$, are resonant values of β [5]. Using (22) and the identity $2mm'/(g\bar{m})$ $=|m|+|m'|-|m-m'|$, $\langle E\rangle_t^{(1,L)}$ reduces to the expression (17) with (19) in Ref. [5]; alternatively, $\langle E\rangle_t^{(1,L)}=Dt$, where D is the coefficient (20) in Ref. [5].

The fluctuating part $\langle E\rangle_t^{(1,F)}$ is given by Eq. (21) with $[t/\bar{m}]$ replaced by $F_t(\bar{m})=[t/\bar{m}]-t/\bar{m}$. Clearly, $\langle E\rangle_t^{(1,F)}=0$ identically only if $F_t(\bar{m})=0$, i.e., $\bar{m}=1$, for all relevant pairs

(m,m'). This will occur in two cases. (a) The potential is monoharmonic, i.e., $V_m=0$ for $|m|>1$. (b) The correlations $C_\beta(m-m')=0$ for $m\ne m'$, corresponding to a uniform (x-independent) distribution (13). In general, using $|F_t(\bar{m})|<1$ and (22), we easily see that $|\langle E\rangle_t^{(1,F)}|<B$ $=(k^2\tilde{C}/2)\Sigma_{mm'>0}mm'|V_mV_{m'}^*|$, where $\tilde{C}$ is an upper bound of $|C_\beta(m)|$ which decays with $|m|$. The quantity B is always finite for a typical, differentiable $V(x)$ with $|V_m|$ decaying faster than $|m|^{-2}$. Thus, $\langle E\rangle_t^{(1,F)}$ is bounded and is therefore much smaller than $\langle E\rangle_t$ for large t. An example of significant fluctuations for small t is considered below.

We now focus on the class of wave packets for which $\tilde{\Psi}(p)$ is piecewise constant on the unit intervals $n\le p<n$ $+1$ for all integers n. This corresponds, by Eq. (4), to a β-independent $\psi_\beta(x)$, i.e., the same $\psi_\beta(x)=\psi(x)$ is associated with all the β-KRs. Since the correlations in (13) are β independent, $C_\beta(m)=C(m)$, one has $\bar{C}_j(m)=C(m)\delta_{j,0}$ in (14). Using the last result in the derivation above of Eq. (21), we see that Eq. (21) can be replaced by an exact formula for *all* times t:

$$\langle E\rangle_t^{(1)}=\frac{k^2}{2}\sum_{mm'>0}mm'V_mV_{m'}^*C(m-m')\left[\frac{t}{\bar{m}}\right]. \quad (23)$$

Also, in Eq. (18), $G_\beta(m)=G_0(m)+\beta C(m)$ for $0\le\beta<1$. Then

$$\bar{G}_j(m)=\int_0^1 d\beta\exp(-2\pi ij\beta)G_\beta(m)$$

$$=\left(G_0(m)+\frac{C(m)}{2}\right)\delta_{j,0}+i\frac{C(m)}{2\pi j}(1-\delta_{j,0}). \quad (24)$$

Inserting $\bar{C}_j(m)=C(m)\delta_{j,0}$ and (24) in (19), we obtain

$$\langle E\rangle_t^{(2)}=-\frac{k}{2\pi l}\sum_{m\ne 0}V_mC(m)\sum_{s=1}^t\frac{(-1)^{lms}}{s}. \quad (25)$$

Let us briefly analyze the exact results (23) and (25). It is clear from the definition of $\bar{m}$ that the fluctuating part $\langle E\rangle_t^{(1,F)}$ of (23) is a periodic or quasiperiodic function of t depending on whether the number of harmonics of the potential is finite or infinite. Concerning (25), we first introduce the notations $W_1=-k/(2\pi l)\Sigma_mV_{2m+1}C(2m+1)$, $W_2=-k/(2\pi l)\Sigma_{m\ne 0}V_{2m}C(2m)$, and $W=W_1+W_2$. A good approximation of (25) for even l is then $\langle E\rangle_t^{(2)}\approx W[\ln(t)+\gamma]$, where $\gamma\approx 0.5772$ is the Euler constant. For odd l, $\langle E\rangle_t^{(2)}$ $\approx W_2[\ln(t)+\gamma]-W_1\ln(2)$. Thus, (25) provides logarithmic corrections to the dominant growth of $\langle E\rangle_t$ given by (23). These corrections vanish if, e.g., the symmetry conditions mentioned after Eq. (19) are satisfied. Then, $\langle E\rangle_t$ $=\langle E\rangle_0+\langle E\rangle_t^{(1)}$, so that the asymptotic behavior (23) of $\langle E\rangle_t$ starts at $t=1$. If, in addition, $\langle E\rangle_t^{(1,F)}=0$ (see the cases in which this occurs in the previous paragraph), $\langle E\rangle_t$ is an exactly linear function of t. See also note [8].

In general, $\langle E\rangle_t^{(2)}$ may be significant relative to $\langle E\rangle_t^{(1)}$ for small t. As a simple example, we consider the case of the two-harmonic potential $V(x)=\cos(x)+\eta\cos(2x)$ with $\psi_\beta(x)=\psi(x)=A[1+\lambda\cos(x)]$, where η and λ are constants and $A=[2\pi(1+\lambda^2/2)]^{-1/2}$ is the normalization factor,

$\int_0^{2\pi} dx |\psi(x)|^2 = 1$. The only nonzero values of V_m and $C(m)$ are those with $|m| \leq 2$ and can be easily calculated from (6) and (13). Inserting them in (23) and in the results above for (25), we get

$$\langle E \rangle_t^{(1)} = k^2 \left(\frac{1}{4}(1 + \eta^2) + \frac{\eta\lambda}{2 + \lambda^2} \right) t + k^2 \frac{2\eta\lambda}{2 + \lambda^2} F_t(2), \quad (26)$$

$$\langle E \rangle_t^{(2)} \approx -\frac{k}{2\pi l} \frac{2\lambda + \eta\lambda^2/2}{2 + \lambda^2} [\ln(t) + \gamma], \quad (27)$$

where $F_t(2) = 0$ or $-1/2$ for t even or odd, respectively, and (27) holds for even l (the case of odd l can be treated similarly and will not be discussed here). Choose $\lambda = \sqrt{2}$, so that $|\lambda|/(2 + \lambda^2)$ assumes its maximal value $\sqrt{2}/4$. Then, for $\eta \approx 1$ and sufficiently small $k \ll 1$, the logarithmic correction (27) is significantly larger than (26) for all times t satisfying $\ln(t)/t \gg 2\pi lk$. If $k > 1$, on the other hand, $\langle E \rangle_t^{(2)}$ is negligible relative to $\langle E \rangle_t^{(1)}$. For $\eta \approx 1$, the fluctuations in (26) have a magnitude of about 40% of that of the coefficient $D \approx \langle E \rangle_t^{(1)}/t$ and may be therefore observed experimentally for not too large t.

We now assume that the initial KP state is an incoherent mixture of plane waves $\exp(ipx)$ with momentum distribution $f(p)$ sufficiently localized in p. At time t, the expectation value of the kinetic energy in the state evolving from a plane wave with $p = n + \beta$ is, from Ref. [5] (with a change in notation),

$$E_t(n, \beta) = \frac{(n + \beta)^2}{2} + k^2 \sum_{m>0} m^2 |V_m|^2 \frac{\sin^2(m\tau_\beta t/2)}{\sin^2(m\tau_\beta/2)}. \quad (28)$$

The MKE of the mixture is then $\langle E_t \rangle = \int_0^1 d\beta \sum_n f(n + \beta) E_t(n, \beta)$. Defining the function $f_0(\beta) = \sum_n f(n + \beta)$, which is periodic in β with Fourier expansion $f_0(\beta) = \sum_j \bar{f}(j) \exp(2\pi i j\beta)$, and using (15), we obtain after a straightforward calculation

$$\langle E_t \rangle = \langle E_0 \rangle + k^2 \sum_{m>0} m^2 |V_m|^2 \sum_{a=1-t}^{t-1} (-1)^{mla} \bar{f}(mla)(t - |a|). \quad (29)$$

Clearly, in the asymptotic time regime $t \gg 1$, (29) always exhibits a linear growth without fluctuations, unlike (21). For a nonuniform mixture [$f_0(\beta) \neq 1$, i.e., $\bar{f}(j) \neq \delta_{j,0}$], this linear growth will be generally preceded by a transient behavior for sufficiently small t.

In conclusion, we have shown that for KP wave packets the asymptotic quantum-resonant evolution of the MKE is typically given by $\langle E \rangle_t \sim Dt + \langle E \rangle_t^{(1,F)}$, where D is the coefficient of linear growth [5] and $\langle E \rangle_t^{(1,F)}$ are bounded fluctuations of a simple number-theoretical nature. For a large class of initial wave packets, $\langle E \rangle_t$ is exactly the superposition of its asymptotic behavior and transient logarithmic corrections which may be negligible for large k or can vanish under some conditions. The asymptotic behavior then starts already at small t, where the fluctuations are most prominent, especially when the magnitude of $\langle E \rangle_t^{(1,F)}$ is comparable to that of D as in the example above. We remark here that the sensitive dependence of D on the harmonics of $V(x)$ and on the initial wave packet, pointed out in Ref. [5], is featured also by $\langle E \rangle_t^{(1,F)}$, as one can clearly see from (21) or (26). The fluctuations in $\langle E \rangle_t$ for small t may be noticeable even when the transient corrections are significant. All the quantum-resonance behaviors for t not too large are expected to be robust under sufficiently small deviations of $\tau/(2\pi)$ from integers and may be therefore observed experimentally.

Since integer values of $\tau/(2\pi)$ correspond to strong quantum regimes, the fluctuations and transient phenomena studied here are basically different in nature from known ones occurring in semiclassical regimes [9–11], e.g., the asymptotic quasiperiodic fluctuations associated with dynamical localization and transient behaviors featuring an approximate coincidence of the classical and quantum evolutions on some initial time interval. It would be interesting, however, if one could establish some correspondence between the general quantum-resonance phenomena in this paper and classical behaviors of related systems, as was done in a particular case in Ref. [4].

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[8] For odd l, the logarithmic corrections will vanish also if $W_2 = 0$, e.g., when $V(x)$ contains only odd harmonics. Then, $\langle E \rangle_t^{(2)} \approx -W_1 \ln(2)$ and, if $\langle E \rangle_t^{(1,F)} = 0$, $\langle E \rangle_t$ will be very close to a linear function of t.

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